Noise Robust Differentiators for Second Derivative Estimation

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This report is about numerical estimation of second derivative by noise robust differentiators. Below we describe method which can be used for uniform as well as for irregular spaced data. In case of uniform spacing method produces exact second derivative for polynomials up to the third degree.

If higher approximation order is required please contact me by e-mail: `pavel@holoborodko.com`. I am open for suggestions and inquiries to build filters for specific needs as well as implementing them in the most optimal way for particular platform.

I would be grateful for information about projects using filters from this report.

Numerical estimation of second derivative by noise robust filters can be written as:

Type	General Formula (Uniform Spacing)	Exact on
Centered	$f''(x_0) \approx \frac{1}{2^{N-3}h^2}\left(s_0 f_0 + \sum_{k=1}^{M} s_k(f_k + f_{-k})\right)$	$1,\ x,\ x^2,\ x^3$
Backward	$f''(x_0) \approx \frac{1}{2^{N-3}h^2}\left(s_0 f_{-M} + \sum_{k=1}^{M} s_k(f_{-M+k} + f_{-M-k})\right)$	$1,\ x,\ x^2$

where $N \geq 5$ is a filter length (must be odd), and $M = (N-1)/2$ is center of symmetry. Coefficients $\{s_k\}_{k=0}^{M}$ are the same for both types of filters. They can be calculated for any N by the simple recursive algorithm ($k = M \ldots 0$):

Algorithm 1: s(k)

if $k > M$ **then** **return** 0;
if $k = M$ **then** **return** 1;
return $[(2N-10)\,\mathtt{s}(k+1) - (N+2k+3)\,\mathtt{s}(k+2)]/(N-2k-1)$

The same coefficients $\{s_k\}$ can be used to approximate second derivative for irregular spaced data.

Type	General Formula (Irregular Spacing)
Centered	$f''(x_0) \approx \frac{1}{2^{N-3}}\left(\sum_{k=1}^{M} \alpha_k(f_k + f_{-k}) - 2f_0\sum_{k=1}^{M}\alpha_k\right)$
Backward	$f''(x_0) \approx \frac{1}{2^{N-3}}\left(\sum_{k=1}^{M} \beta_k(f_{-M+k} + f_{-M-k}) - 2f_{-M}\sum_{k=1}^{M}\beta_k\right)$

where

$$\alpha_k = \frac{4k^2 s_k}{(x_k - x_{-k})^2}, \qquad \beta_k = \frac{4k^2 s_k}{(x_{-M+k} - x_{-M-k})^2}$$