One-Sided Differentiators

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One sided/backward/without time delay/causal numerical differentiation is special method for derivative estimation which uses only past values of a function. Such differentiators can be written as

$$f'(x_i) \approx \frac{1}{h}\sum_{k=0}^{N} c_k f_{i-k}$$

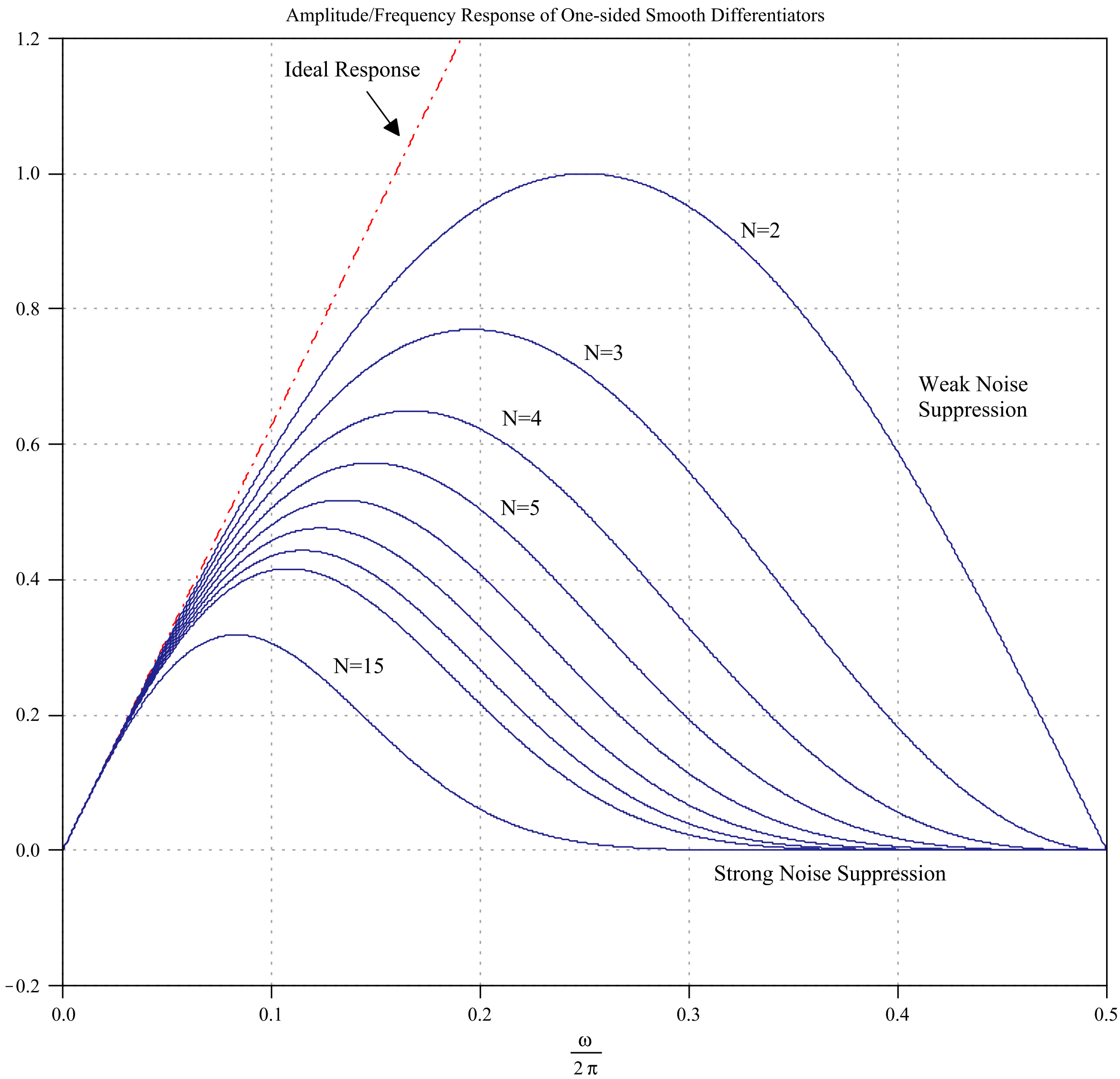
where array $\{c_k\}$ is coefficients, h sampling step and $f_{i-k} = f(x_i - kh)$.

In this report I present two types of one sided differentiators. First combines numerical derivative estimation and guaranteed noise suppression towards upper bound of Nyquist interval. Second is intended to find derivative exactly for the polynomials 1, x, x^2 and suppress noise in minimax sense (differ from Savitzky-Golay filters by stronger noise suppression near $\omega = \pi$).

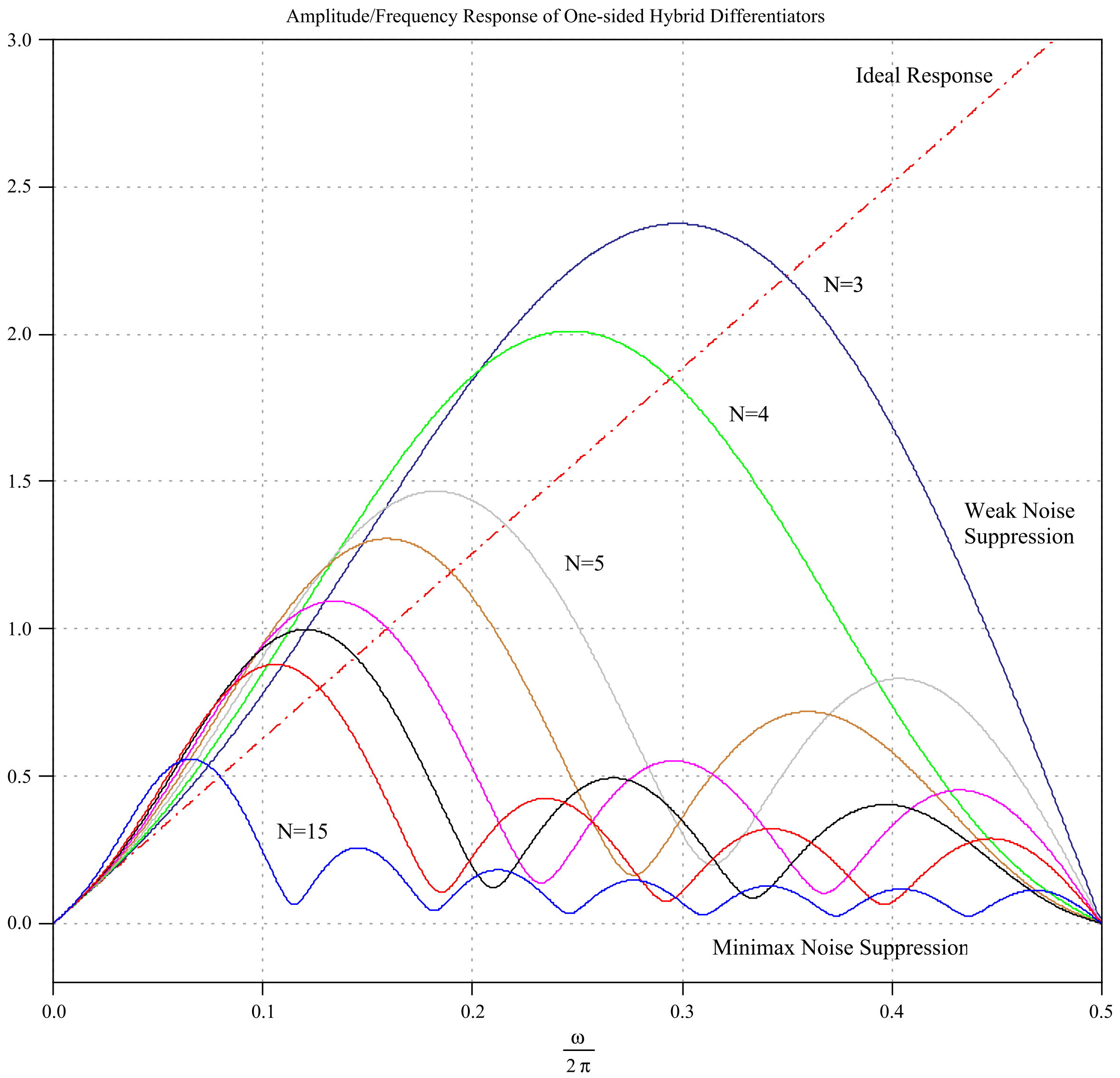
Every particular task requires distinct differentiator with specific properties. I do not try to cover all possible cases in this report. However I am open for suggestions and inquiries to build filters for specific needs as well as implementing them in the most optimal way for particular platform.

Please contact me by e-mail `pavel@holoborodko.com` or visit my site `http://www.holoborodko.com/pavel/` for more information. Also I would be grateful for information about projects using filters from this report.

N	One Sided Smooth Differentiators (exact on 1, x)
2	$\frac{1}{2h}\left(f_i - f_{i-2}\right)$
3	$\frac{1}{4h}\left(f_i + f_{i-1} - f_{i-2} - f_{i-3}\right)$
4	$\frac{1}{8h}\left(f_i + 2\,f_{i-1} - 2\,f_{i-3} - f_{i-4}\right)$
5	$\frac{1}{16h}\left(f_i + 3\,f_{i-1} + 2\,f_{i-2} - 2\,f_{i-3} - 3\,f_{i-4} - f_{i-5}\right)$
6	$\frac{1}{32h}\left(f_i + 4\,f_{i-1} + 5\,f_{i-2} - 5\,f_{i-4} - 4\,f_{i-5} - f_{i-6}\right)$
7	$\frac{1}{64h}\left(f_i + 5\,f_{i-1} + 9\,f_{i-2} + 5\,f_{i-3} - 5\,f_{i-4} - 9\,f_{i-5} - 5\,f_{i-6} - f_{i-7}\right)$
8	$\frac{1}{128h}\left(f_i + 6\,f_{i-1} + 14\,f_{i-2} + 14\,f_{i-3} - 14\,f_{i-5} - 14\,f_{i-6} - 6\,f_{i-7} - f_{i-8}\right)$
9	$\frac{1}{256h}(f_i + 7\,f_{i-1} + 20\,f_{i-2} + 28\,f_{i-3} + 14\,f_{i-4}$ $-14\,f_{i-5} - 28\,f_{i-6} - 20\,f_{i-7} - 7\,f_{i-8} - f_{i-9})$
10	$\frac{1}{512h}(f_i + 8\,f_{i-1} + 27\,f_{i-2} + 48\,f_{i-3} + 42\,f_{i-4}$ $-42\,f_{i-6} - 48\,f_{i-7} - 27\,f_{i-8} - 8\,f_{i-9} - f_{i-10})$
15	$\frac{1}{16384h}(f_i + 13\,f_{i-1} + 77\,f_{i-2} + 273\,f_{i-3} + 637\,f_{i-4} + 1001\,f_{i-5}$ $+1001\,f_{i-6} + 429\,f_{i-7} - 429\,f_{i-8} - 1001\,f_{i-9} - 1001\,f_{i-10}$ $-637\,f_{i-11} - 273\,f_{i-12} - 77\,f_{i-13} - 13\,f_{i-14} - f_{i-15})$


Amplitude/Frequency Response of One-sided Smooth Differentiators
Ideal Response
N=2
N=3
N=4
N=5
N=15
Weak Noise Suppression
Strong Noise Suppression
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0.0
0.1
0.2
0.3
0.4
0.5
ω
2 π

N	One Sided Hybrid Differentiators (exact on 1, x, x^2)
3	$\frac{1}{2h}\left(2\,f_i - f_{i-1} - 2\,f_{i-2} + f_{i-3}\right)$
4	$\frac{1}{10h}\left(7\,f_i + f_{i-1} - 10\,f_{i-2} - f_{i-3} + 3\,f_{i-4}\right)$
5	$\frac{1}{28h}\left(16\,f_i + f_{i-1} - 10\,f_{i-2} - 10\,f_{i-3} - 6\,f_{i-4} + 9\,f_{i-5}\right)$
6	$\frac{1}{28h}\left(12\,f_i + 5\,f_{i-1} - 8\,f_{i-2} - 6\,f_{i-3} - 10\,f_{i-4} + f_{i-5} + 6\,f_{i-6}\right)$
7	$\frac{1}{60h}\left(22\,f_i + 7\,f_{i-1} - 6\,f_{i-2} - 11\,f_{i-3} - 14\,f_{i-4} - 9\,f_{i-5} - 2\,f_{i-6} + 13\,f_{i-7}\right)$
8	$\frac{1}{180h}\left(52\,f_i + 29\,f_{i-1} - 14\,f_{i-2} - 17\,f_{i-3} - 40\,f_{i-4} - 23\,f_{i-5} - 26\,f_{i-6} + 11\,f_{i-7} + 28\,f_{i-8}\right)$
9	$\frac{1}{220h}\left(56\,f_i + 26\,f_{i-1} - 2\,f_{i-2} - 17\,f_{i-3} - 30\,f_{i-4} - 30\,f_{i-5} - 28\,f_{i-6} - 13\,f_{i-7} + 4\,f_{i-8} + 34\,f_{i-9}\right)$
10	$\frac{1}{1540h}\left(320\,f_i + 206\,f_{i-1} - 8\,f_{i-2} - 47\,f_{i-3} - 186\,f_{i-4} - 150\,f_{i-5} - 214\,f_{i-6} - 103\,f_{i-7} - 92\,f_{i-8} + 94\,f_{i-9} + 180\,f_{i-10}\right)$
15	$\frac{1}{2856h}\left(322\,f_i + 217\,f_{i-1} + 110\,f_{i-2} + 35\,f_{i-3} - 42\,f_{i-4} - 87\,f_{i-5} - 134\,f_{i-6} - 149\,f_{i-7} - 166\,f_{i-8} - 151\,f_{i-9} - 138\,f_{i-10} - 93\,f_{i-11} - 50\,f_{i-12} + 25\,f_{i-13} + 98\,f_{i-14} + 203\,f_{i-15}\right)$


Amplitude/Frequency Response of One-sided Hybrid Differentiators
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Ideal Response
N=3
N=4
Weak Noise
Suppression
N=5
N=15
Minimax Noise Suppression
0.0
0.1
0.2
0.3
0.4
0.5
ω
2 π